CONDITIONS OF TOROIDAL EQUILIBRIUM OF A MOVING PLASMA

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Necessary conditions for stationary plasma flow in a magnetic field of toroidal geometry are derived in a magnetohydrodynamic approximation.

It is well known [1] that toroidal equilibrium of a quiescent plasma requires that the azimuthal component H_ω of the magnetic field corresponding to the angle of the short circuit of the torus be nonzero,

$$H_\omega \neq 0. \tag{1}$$

One possible method of studying Eq. (1) is to find the stationary solutions of magnetic hydrodynamics equations for a stationary medium to a first approximation relative to the torus parameter $\alpha = a/R$ (a and R are the radii of the short and long circuits of the torus), which may be considered small. The resulting solutions have meaning only when Eq. (1) holds, since H_ω (more precisely, $H_{0\omega}$, which corresponds to the zeroth approximation $\alpha = 0$) occurs in the denominator of the resulting expressions. Conditions of toroidal equilibrium for a moving medium may be analogously obtained.

In the magnetohydrodynamic approximation the equations describing steady motion have the form

$$\rho(\mathbf{v}\nabla)\mathbf{v} + \nabla p - \frac{1}{4\pi}[\mathrm{rot}\,\mathbf{H}, \mathbf{H}] = 0; \quad \mathrm{rot}\,[\mathbf{v}\mathbf{H}] = 0;$$
$$\mathrm{div}\,\rho\mathbf{v} = 0; \quad \mathrm{div}\,\mathbf{H} = 0; \quad (\mathbf{v}\nabla)S = 0, \tag{2}$$

where ρ, p, **v**, and S are density, pressure, velocity, and entropy of the plasma, and **H** is magnetic field strength.

Let us use a toroidal coordinate system $\{r, \omega, \theta\}$, where r is the distance to the circular axis of the toroid and θ and ω are the angles of the large (about the toroid axis) and small circuits (about the circular axis). We introduce the dimensionless variable $\xi = r/R$, writing the system (2) in the form

$$\rho\left(v_r\frac{\partial v_r}{\partial\xi} + \frac{v_\omega}{\xi}\frac{\partial v_r}{\partial\omega} - \frac{\alpha v_\theta^2\cos\omega}{1+\alpha\xi\cos\omega} - \frac{v^2\omega}{\xi}\right) + \frac{\partial}{\partial\xi}\left(p + \frac{H_\theta^2 + H_\omega^2}{8\pi}\right) - \frac{1}{4\pi}\left(\frac{H_\omega}{\xi}\frac{\partial H_r}{\partial\omega} - \frac{\alpha H_\theta^2\cos\omega}{1+\alpha\xi\cos\omega} - \frac{H_\omega^2}{\xi}\right) = 0; \tag{3}$$

$$\rho\left(v_r\frac{\partial v_\omega}{\partial\xi} + \frac{v_\omega}{\xi}\frac{\partial v_\omega}{\partial\omega} + \frac{\alpha v_\theta^2\sin\omega}{1+\alpha\xi\cos\omega} + \frac{v_r v_\omega}{\xi}\right) + \frac{1}{\xi}\frac{\partial}{\partial\omega}\left(p + \frac{H_r^2 + H_\omega^2}{8\pi}\right) - \frac{1}{4\pi}\left(H_r\frac{\partial H_\omega}{\partial\xi} + \frac{\alpha H_\theta^2\sin\omega}{1+\alpha\xi\cos\omega} + \frac{H_r H_\omega}{\xi}\right) = 0;$$

$$\rho\left(v_r\frac{\partial v_\theta}{\partial\xi} + \frac{v_\omega}{\xi}\frac{\partial v_\theta}{\partial\omega} + \frac{\alpha v_\theta v_r\cos\omega}{1+\alpha\xi\cos\omega} - \frac{\alpha v_\theta v_\omega\sin\omega}{1+\alpha\xi\cos\omega}\right) - \frac{1}{4\pi}\left[H_r\frac{\partial H_\theta}{\partial\xi} + \frac{H_w}{\xi}\frac{\partial H_\theta}{\partial\omega} + \frac{\alpha H_\theta}{1+\alpha\xi\cos\omega}(H_\rho\cos\omega - H_\omega\sin\omega)\right] = 0;$$

$$\frac{\partial}{\partial\omega}[(1+\alpha\xi\cos\omega)(v_\omega H_r - v_r H_\omega)] = 0; \quad \frac{\partial}{\partial\omega}(v_\theta H_\omega - v_\omega H_\theta) - \frac{\partial}{\partial\xi}[\xi(v_r H_\theta - v_\theta H_r)] = 0;$$

$$(1+\alpha\xi\cos\omega)\frac{\partial}{\partial\xi}(v_\omega H_r - v_r H_\omega) + \alpha\cos\omega(v_\omega H_r - v_r H_\omega) = 0;$$

$$\frac{\partial}{\partial\xi}[\xi(1+\alpha\xi\cos\omega)\rho v_r] + \frac{\partial}{\partial\omega}[(1+\alpha\xi\cos\omega)\rho v_\omega] + \alpha\frac{\partial}{\partial\theta}(\xi\rho v_\theta) = 0;$$

$$\frac{\partial}{\partial\xi}[\xi(1+\alpha\xi\cos\omega)H_r] + \frac{\partial}{\partial\omega}[(1+\alpha\xi\cos\omega)H_\omega] + \alpha\frac{\partial}{\partial\theta}(\xi H_\theta) = 0;$$

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$$v_r \frac{\partial p}{\partial \xi} + \frac{v_\omega}{\xi} \frac{\partial p}{\partial \omega} - \frac{\gamma p}{\rho} \left(v_r \frac{\partial \rho}{\partial \xi} + \frac{v_\omega}{\xi} \frac{\partial \rho}{\partial \omega} \right) = 0.$$

Here γ is the adiabatic index.

If we assume that the parameter α is small, we can represent the desired variables f_i in the form $f_i = f_{0i} + f_{1i}$, where f_{0i} corresponds to the zeroth and f_{1i} to the first, approximation relative to α. Let us assume that no dependence on θ and ω exists in the zeroth approximation,

$$\frac{\partial f_{0i}}{\partial \theta} = \frac{\partial f_{0i}}{\partial \omega} = 0.$$

Then the system (3) reduces, in the zeroth approximation with respect to α, to the single equation

$$\frac{\partial}{\partial \xi} \left(p_0 + \frac{H_{0\theta}^2 + H_{0\omega}^2}{8\pi} \right) + \frac{H_{0\omega}^2}{4\pi\xi} - \frac{\rho_0 v_{0\omega}^2}{\xi} = 0, \tag{4}$$

and in a first approximation, to the system of equations

$$\rho_0 \left(\frac{v_{0\omega}}{\xi} \frac{\partial v_{1r}}{\partial \omega} - \alpha v_{0\theta}^2 \cos\omega - \frac{2 v_{0\omega} v_{1\omega}}{\xi} \right) - \frac{\rho_1 v_{0\omega}^2}{\xi} + \frac{\partial}{\partial \xi} \left(p_1 + \frac{H_{0\theta} H_{1\theta} + H_{0\omega} H_{1\omega}}{4\pi} \right) - \frac{1}{4\pi} \left(\frac{H_{0\omega}}{\xi} \frac{\partial H_{1r}}{\partial \omega} - \alpha H_{0\theta}^2 \cos\omega - \frac{2 H_{0\omega} H_{1\omega}}{\xi} \right) = 0; \tag{5}$$

$$\rho_0 \left(v_{1r} \frac{\partial v_{0\omega}}{\partial \xi} - \frac{v_{0\omega}}{\xi} \frac{\partial v_{1\omega}}{\partial \omega} + \alpha v_{0\theta}^2 \sin\omega + \frac{v_{1r} v_{0\omega}}{\xi} \right) + \frac{1}{\xi} \frac{\partial}{\partial \omega} \left(p_1 + \frac{H_{0\theta} H_{1\theta}}{4\pi} \right) - \frac{1}{4\pi} \left(H_{1r} \frac{\partial H_{0\omega}}{\partial \xi} + \frac{H_{1r} H_{0\omega}}{\xi} + \alpha H_{0\theta}^2 \sin\omega \right) = 0; \tag{6}$$

$$\rho_0 \left(v_{1r} \frac{\partial v_{0\theta}}{\partial \xi} + \frac{v_{0\omega}}{\xi} \frac{\partial v_{1\theta}}{\partial \omega} - \alpha v_{0\theta} v_{0\omega} \sin\omega \right) - \frac{1}{4\pi} \left(H_{1r} \frac{\partial H_{0\theta}}{\partial \xi} + \frac{H_{0\omega}}{\xi} \frac{\partial H_{1\theta}}{\partial \omega} - \alpha H_{0\theta} H_{0\omega} \sin\omega \right) = 0; \tag{7}$$

$$\frac{\partial}{\partial \omega} \left(v_{0\omega} H_{1r} - v_{1r} H_{0\omega} \right) = 0; \tag{8}$$

$$\frac{\partial}{\partial \omega} \left(v_{0\theta} H_{1\omega} + v_{1\theta} H_{0\omega} - v_{0\omega} H_{1\theta} - v_{1\omega} H_{0\theta} \right) - \frac{\partial}{\partial \xi} \left[\xi \left(v_{1r} H_{0\theta} - v_{0\theta} H_{1r} \right) \right] = 0; \tag{9}$$

$$\frac{\partial}{\partial \xi} \left(\xi \rho_0 v_{1r} \right) + \frac{\partial}{\partial \omega} \left(\rho_1 v_{0\omega} + \rho_0 v_{1\omega} \right) - \alpha \xi \rho_0 v_{0\omega} \sin\omega = 0; \tag{10}$$

$$\frac{\partial}{\partial \xi} \left(\xi H_{1r} \right) - \alpha \xi H_{0\omega} \sin\omega + \frac{\partial H_{1\omega}}{\partial \omega} = 0; \tag{11}$$

$$v_{1r} \frac{\partial p_0}{\partial \xi} + \frac{v_{0\omega}}{\xi} \frac{\partial p_1}{\partial \omega} - \frac{\gamma p_0}{\rho_0} \left(v_{1r} \frac{\partial \rho_0}{\partial \xi} + \frac{v_{0\omega}}{\xi} \frac{\partial \rho_1}{\partial \omega} \right) = 0. \tag{12}$$

We find the desired variables f_{1i} using the system of equations (5)-(12), bearing in mind that f_{0i} satisfies Eq. (4). We express the remaining unknowns in terms of H_{1r} and $H^*_{1r} = \int H_{1r} d\omega$ for this purpose by means of Eqs. (6)-(12) and, substituting the unknowns in Eqs. (5), we obtain an equation to find H_{1r}.

We first express the unknowns in Eqs. (7) and (8)-(12) in terms of p_1, H_{1r}, and H^*_{1r}. We find that

$$v_{1r} = \frac{v_{0\omega}}{H_{0\omega}} H_{1r}; \tag{13}$$

$$\rho_1 = \frac{\rho_0}{\gamma p_0} p_1 + \frac{\xi}{H_{0\omega}} \left(\frac{\rho_0}{\gamma p_0} \frac{\partial \rho_0}{\partial \xi} - \frac{\partial \rho_0}{\partial \xi} \right) H^*_{1r}; \tag{14}$$

$$v_{1\omega} = \frac{\partial}{\partial \xi} \left(\xi \frac{v_{0\omega}}{H_{0\omega}} H^*_{1r} \right) - \frac{v_{0\omega} \xi}{\gamma H_{0\omega} p_0} \frac{\partial p_0}{\partial \xi} H^*_{1r} - \frac{v_{0\omega}}{\gamma p_0} p_1 - \alpha \xi v_{0\omega} \cos\omega; \tag{15}$$

$$H_{1\omega} = -\frac{\partial}{\partial \xi} \left(\xi H^*_{1r} \right) - \alpha \xi \cos\omega \, H_{0\omega}; \tag{16}$$

$$v_{1\theta} = \frac{1}{H_{0\omega}^2 - 4\pi\rho_0 v_{0\omega}^2} \left\{ \xi H^*_{1r} \left[\frac{\partial v_{0\theta}}{\partial \xi} \frac{1}{H_{0\omega}} \left(4\pi\rho_0 v_{0\omega}^2 - H_{0\omega}^2 \right) - \frac{H_{0\theta} v_{0\omega}}{\gamma p_0} \frac{\partial p_0}{\partial \xi} \right] - \frac{v_{0\omega} H_{0\omega} H_{0\theta}}{\gamma p_0} p_1 + \alpha \xi \left[v_{0\theta} \left(H_{0\omega}^2 + 4\pi\rho_0 v_{0\omega}^2 \right) - 2 v_{0\omega} H_{0\theta} H_{0\omega} \right] \cos\omega \right\}; \tag{17}$$

$$H_{1\theta} = \frac{1}{H_{0\omega}^2 - 4\pi\rho_0 v_{0\omega}^2} \left\{ \xi H^*_{1r} \left[\frac{1}{H_{0\omega}} \frac{\partial H_{0\theta}}{\partial \xi} \left(4\pi\rho_0 v_{0\omega}^2 - H_{0\omega}^2 \right) \right. \right.$$

$$-\frac{4\pi\rho_0 v_{0\omega}^2}{\gamma p_0}\frac{\partial p_0}{\partial\xi}\frac{H_{0\theta}}{H_{0\omega}}\Bigg]-\frac{4\pi\rho_0 v_{0\omega}^2 H_{0\theta}}{\gamma p_0}p_1-\alpha\xi\left[H_{0\theta}\left(H_{0\omega}^2+4\pi\rho_0 v_{0\omega}^2\right)-8\pi\rho_0 v_{0\omega}v_{0\theta}H_{0\omega}\right]\cos\omega\Big\}. \quad (18)$$

Integrating Eq. (6), over ω and substituting in it Eqs. (13)-(18), we find p_1 in terms of H^*_{1r},

$$p_1=\frac{\rho_0 v_{0\omega}^2}{F}\frac{\partial}{\partial\xi}\left(\frac{\xi H_{1r}^*}{H_{0\omega}}\right)-\left(\frac{\xi H_{1r}^*}{H_{0\omega}}\right)\frac{\partial p_0}{\partial\xi}+\alpha\xi\frac{G}{F}\cos\omega, \quad (19)$$

$$F=1-\frac{v_{0\omega}^2}{c_0^2}\left(1+\frac{H_{0\theta}^2}{H_{0\omega}^2-4\pi\rho_0 v_{0\omega}^2}\right),\ c_0^2=\frac{\gamma p_0}{\rho_0},$$

$$G=\rho_0\left(v_{0\omega}^2+v_{0\theta}^2\right)-\frac{H_{0\theta}^2}{4\pi}-\frac{1}{\left(H_{0\omega}^2-4\pi\rho_0 v_{0\omega}^2\right)}\left[2\rho_0 v_{0\omega}v_{0\theta}H_{0\omega}H_{0\theta}-\frac{H_{0\theta}^2}{4\pi}\left(4\pi\rho_0 v_{0\omega}^2+H_{0\omega}^2\right)\right].$$

We obtain an equation for finding H^*_{1r} by substituting Eq. (19) in Eqs. (13)-(18), subsequently substituting Eqs. (13)-(19) in Eqs. (5), since $H^*_{1r}=y(\xi)\cos\omega$ (i.e., $H_{1r}=-y\sin\omega$)

$$\frac{d}{d\xi}(Az)+Bz+\alpha\Phi=0,\ z=\frac{d}{d\xi}\left(\frac{y\xi}{H_{0\omega}}\right), \quad (20)$$

$$A=\frac{\rho_0}{F}\left[v_{0\omega}^2\left(1+\frac{c_A^2}{c_0^2}\right)-c_{A\omega}^2\right],$$

$$c_{A\omega}^2=\frac{H_{0\omega}^2}{4\pi\rho_0},\ c_A^2=\frac{H_{0\omega}^2+H_{0\theta}^2}{4\pi\rho_0},$$

$$B=\frac{1}{\xi}\left(\frac{\rho_0 v_{0\omega}^4}{c_0^2F}+\rho_0 v_{0\omega}^2-\frac{H_{0\omega}^2}{4\pi}\right),$$

$$\Phi=\frac{d}{d\xi}\left\{\xi\left[\frac{G}{F}-\frac{H_{0\omega}^2}{4\pi}+\frac{1}{H_{0\omega}^2-4\pi\rho_0 v_{0\omega}^2}\left(2\rho_0 v_{0\omega}v_{0\theta}H_{0\omega}H_{0\theta}\right.\right.\right.$$

$$\left.\left.\left.-\frac{H_{0\theta}^2}{4\pi}\left(H_{0\omega}^2+4\pi\rho_0 v_{0\omega}^2\right)-\frac{v_{0\omega}^2}{c_0^2}\frac{H_{0\theta}^2}{F}\right]\right\}+\left[\frac{v_{0\omega}^2}{c_0^2}\frac{G}{F}+2\rho_0 v_{0\omega}^2-\rho_0 v_{0\theta}^2-\frac{1}{4\pi}\left(2H_{0\omega}^2-H_{0\theta}^2\right)\right].$$

We find, by solving Eq. (20), that

$$y(\xi)=-\frac{H_{0\omega}}{\xi}\int e^{-\int\frac{B}{A}d\xi}\left[-\frac{\alpha}{A}\int\Phi e^{\int\frac{B}{A}d\xi}d\xi+\frac{C_1}{A}\right]d\xi+C_2, \quad (21)$$

where C_1 and C_2 are constants of integration.

Let us consider the singular points of these solutions, at which the unknowns f_{1i} become infinite. Since such infinities indicate that the solution of the initial system of equations cannot be solved, we eliminate these singular points, obtaining necessary conditions for stationary flow of a plasma in a toroid.

Equations (21) imply that A can never be zero, so that

$$v_{0\omega}^2\neq c_{A\omega}^2\left(1+c_A^2/c_0^2\right)^{-1}, \quad (22)$$

and Eq. (19) implies that $f\neq 0$, so that

$$v_{0\omega}\neq c_0\left[\frac{1}{2}\left(1+c_A^2/c_0^2\right)\pm\sqrt{\frac{1}{4}\left(1+c_A^2/c_0^2\right)^2-c_{A\omega}^2/c_0^2}\,\right]. \quad (23)$$

Equations (22) and (23) are necessary conditions for the possibility of stationary plasma flow in a toroid to a magnetohydrodynamic approximation.

If the denominators of the right sides of Eqs. (17) and (18) vanish when $v_{0\omega}=c_{A\omega}$, this will not lead to the appearance of infinities, since the vanishing factors are eliminated when p_1 from Eq. (19) is substituted in these equations.

According to Eq. (19), when $(H_{0\omega}^2-4\pi\rho_0 v_{0\omega}^2)\to 0$,

$$p_1+\frac{\xi H_{1r}^*}{H_{0\omega}}\frac{\partial p_0}{\partial\xi}=\frac{c_0^2}{4\pi v_{0\omega}^2 H_{0\theta}^2}\left[8\pi\rho_0 v_{0\omega}v_{0\theta}H_{0\omega}H_{0\theta}-H_{0\theta}^2\left(4\pi\rho_0 v_{0\omega}^2+H_{0\omega}^2\right)\right]\alpha\xi\cos\omega. \quad (24)$$

The condition (24) in the cylindrical case ($\alpha=0$) takes the form, according to Eq. (13),

$$v_{1r}\frac{\partial p_0}{\partial \xi}+\frac{v_{0\omega}}{\xi}\frac{\partial p_1}{\partial \omega}=0,$$

which in this case ($v_{0r}=0$, $\partial p_0/\partial\xi \neq 0$) is equivalent to the requirement that pressure along the projection of the streamline on a plane perpendicular to the cylinder axis be constant.

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INDUCED FLUCTUATIONS OF THE INTENSITY OF RADIATION EXCITED BY AN ELECTRON BEAM IN A FLOW OF RAREFIED GAS WITH CLUSTERS

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The presence of fluctuations in the intensity of radiation excited by electrons in a jet core was discovered using an electron beam and by studying the interaction of two low-density supersonic CO_2 flows. The nature of the variation of the frequency and amplitude of the surges within the jet core as well as the region within which the surges exist depend on the parameters of the flow retardation.

The use of an electron beam for diagnostics of flows of rarefied gas has become widespread in experimental gasdynamic studies because of the fact that these methods result in quantitative data on gas density and the concentration of components and their energy states both in the quiescent and in the moving gas [1, 2]. Measurements were based on the ability to establish a unique relation between the intensity and nature of the spectrum excited by the electron beam, and the state of the gas. In this work, low-frequency radiation fluctuations in the zone of an electron beam used for probing interacting flows of rarefied gas containing clusters are studied.

This phenomenon has been studied for the interaction of two CO_2 slipstreams in a vacuum chamber. A gas-driven source with a supersonic nozzle (critical cross-section diameter $d_*=0.53$ mm, section diameter $d=1.22$ mm) or with a sonic nozzle ($d_*=0.33$ mm) was mounted in the flow field at a distance of 1120 mm behind the nozzle with section diameter 100 mm and geometric Mach number $M_1\approx 8$. The dimensions of the source were chosen to be 10 times several mean free paths, in order that the gas of the external slipstream streamlining the jet not be substantially influenced. The parameters of these flow conditions are indicated in

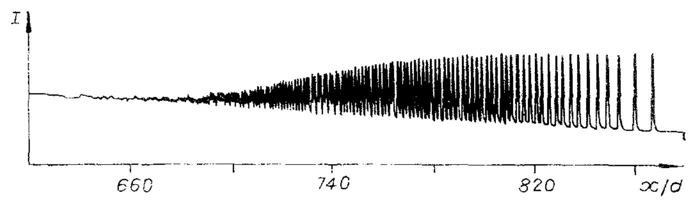

Fig. 1

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